SIMPLIFIED GUITAR BRIDGE MODEL FOR THE DISPLACEMENT WAVE REPRESENTATION IN DIGITAL WAVEGUIDES

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ABSTRACT

In this paper, we present a simplified model for the string-bridge interaction in guitars or other string instruments simulated by digital waveguides. The bridge model is devised for the displacement wave representation in order to be integrated with other models for string interactions with the player and with other parts of the instrument, whose simulation and implementation is easier in this representation. The model is based on a multiplierless scattering matrix representing the string-bridge interaction. Although not completely physically inspired, we show that this junction is sufficiently general to accommodate a variety of transfer functions under the sole requirement of passivity and avoids integration constants mismatch when the bridge is in turn modeled by a digital waveguide. The model is completed with simple methods to introduce horizontal and vertical polarizations of the string displacement and sympathetic vibrations of other strings. The aim of this paper is not to provide the most general methods for sound synthesis of guitar but, rather, to point at low computational cost and scalable solutions suitable for real-time implementations where the synthesizer is running together with several other audio applications.

1. INTRODUCTION

Although more general frameworks, e.g. based on time domain finite differences (TDFD) schemes [1] on non-uniform space-time stencils have been proposed, the Digital Waveguide (DW) paradigm [2] is appealing for the physically inspired synthesis of musical instruments, including guitar, which is the object of this paper. However, the basic propagating wave model based on the ideal string model, which is implemented in the DW with simple delay lines and pure reflections at the bridge and nut, is clearly not sufficient. To achieve realistic synthesis the simple DW technique must be extended to include dispersive propagation due to string bending stiffness, accurate modeling of the bridge and the nut, as well as usable models for the interaction of the player with the instrument and for collisions of the string with other parts of the instruments.

In a previous paper [3] we proposed a simple model of the plucking action exerted by the player on the string, which is demonstrated in the PluckSynth plugin, which can be freely downloaded at http://staffwww.itn.liu.se/∼giaev/soundexamples.html.

New models for the collision of the string with the fretboard, for the production of harmonics and for the imperfect clamping of the fingers on the fretboard are proposed in a forthcoming paper [4]. All these models were devised using displacement waves to represent the solution. In particular, the collision model requires continuous testing of the string displacement to detect the instants in which the string comes in contact with the neck or other obstacles. This is most efficiently performed in the displacement traveling wave representation, without the need for time integration of the solution.

Several approaches have been proposed for the accurate modeling of the guitar bridge in classical or acoustic guitars [5, 6, 7]. As in most models, the ultimate calibration requires accurate measures of the admittance functions of the bridge in both horizontal and vertical string polarizations, together with cross-coupling of these modes. In the measurements it is quite difficult to isolate the effect of the bridge from the influence of the guitar body. The analysis is complicated by the fact that the bridge rests on a vibrating soundboard, which is a complex mechanical system deeply influencing the dynamics of the bridge in acoustic instruments. This system is best described by modal analysis rather than digital waveguide meshes. In electric guitars the bridge rests on considerably more rigid supports. While the solid or partly hollow body still vibrates, its effect on the bridge dynamics is less prominent.

In this paper we propose a simple approach for modeling the influence of the bridge on the oscillations of the string based on a multiplierless scattering matrix to model the generic interaction, together with a transfer function modeling the termination. Although our method is worked out in the displacement wave representation of the solution, its principles can be translated to other representations. As the general modeling task is quite overwhelming, we will confine ourselves, for the time being, to the model of the considerably simpler bridge found in electric guitars, whether archtop or flattop. In other words, we disregard the important details of the effect of the soundboard and radiation on the string oscillations. We show that, in spite of its simplicity, the scattering element preserves passivity and its use does not restrict the class of termination transfer functions. The model is completed with methods to introduce horizontal and vertical polarizations of the string displacement and sympathetic vibrations from other strings.

The paper is organized as follows. In Section 2 we derive a model for the string-bridge connection. In Section 3 we include considerations on the simulation of horizontal and vertical string polarization modes within the model. In Section 4 we comment on some of the results obtained. Finally, in Section 5 we draw our conclusions.

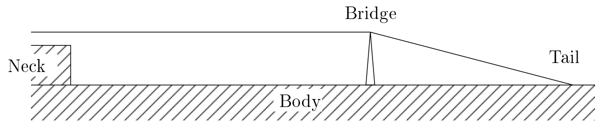

Figure 1: *Diagram of the string-bridge geometry.*

2. A MODEL FOR THE CONTACT OF THE STRING WITH BOUNDARY SYSTEMS

In this section we derive a scattering junction suitable to model a typical bridge of electric guitars or other contacts of the string with the nut and the frets. While bridge modeling is usually accomplished by considering wave impedances and admittances of the bridge or the nut, these are most conveniently formalized in the velocity waves representation of the solution [7]. While it is true that most results can be carried over to the displacement wave representation, the results depend on integration constants. It is well known that D'Alembert's solution of the wave equation in terms of the sum of traveling waves is invariant by the addition of opposite sign constants to the two wave variables. Especially when the string and the bridge interaction are both modeled by means of waveguides, arbitrary constants mismatch on both sides can cause amplitude jumps to the wave variables. These jumps are not revealed in the solution as they cancel out. However, when the interaction concerns one of the two wave variables in one waveguide with another one in the other waveguide, constant mismatch can cause problems. The same is true for interactions of the string with other parts of the instrument as the amplitude jumps travel along the string.

Motivated by a "clean," mismatch-free, modeling in the displacement wave variables, our starting point is the inclusion of a generic system attached to each end of the DW to represent the contact of the string with the termination. The generic system that we propose is described by a multiplierless scattering matrix, whose entries are $\pm 1/2$. Although this matrix is not strictly derived from a specific physical model, it includes all the necessary features so that, in a linear model, the transfer function of the bridge termination regulating how the incident transversal wave is reflected by the string-bridge junction back into the string, can be arbitrarily adjusted or fitted to measurements, under the sole requirement of passivity.

In most guitars, the string is pulled from head to tail and "sees" the nut and the bridge as obstacles on its way. As shown in Fig. 1, at these points the string is slightly bent toward the body. Since the vertical component of the static tension of the string is increased by bending, it holds the string pressed against the bridge and the nut. A similar bending occurs when a finger presses the string against the fretboard in the proximity of a fret. A detail of the bridge and the tailpiece in a semiacoustic guitar is shown in Fig. 2. The active portion of the string spanning from bridge to nut or fret is free to vibrate. The passive part of the string spanning from bridge to tail piece is strongly damped, although in some guitars a fairly long portion ($\approx$ 10 cm) of the string lying between the string and the tailpiece is only moderately damped.

Figure 2: *The bridge and tailpiece in a Gibson ES335 semiacoustic guitar.*

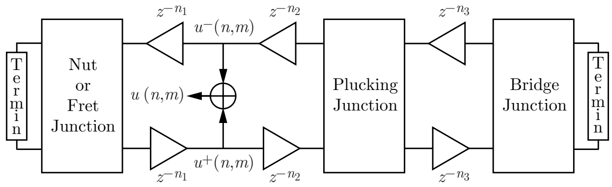

Figure 3: *Overview of the digital waveguide layout.*

2.1. Digital Waveguides and Exceptional Points

For a flexible string in a linear regime, the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where $u = u(x,t)$ denotes the deformation of the string with respect to the equilibrium position holds everywhere, except at special points, which include the terminations and the segment where the player plucks the string. In (1), $c = \sqrt{K_0/\mu}$ is the propagation velocity, μ is the linear mass density of the string and K_0 is the string tension that is assumed to be constant. D'Alembert's solution of (1) can be written in form as the sum

$$u(x,t) = u^-(x,t) + u^+(x,t) \tag{2}$$

where $u^-(x,t) = f_l(t + x/c)$ is a regressive (left going) propagating wave and $u^+(x,t) = f_r(t - x/c)$ is a progressive (right going) propagating wave.

Wave propagation is simulated in a DW by space-time sampling the solution $u(x,t)$. Given a temporal sampling interval T, the spatial sampling interval is chosen as $X = cT$, which simplifies the form of the discrete traveling wave solution. This allows us to compute propagation in terms of two delay lines consisting of chains of elementary delays, constituting the two rails of a DW. At any given location, the deformation of the string is computed as the sum of the contents of the upper and lower rails. The complete scheme of a DW including exceptional points, such as the nut or fret, the bridge and the plucking zone is shown in Fig. 3. Rather than modeling the exceptional points of the string as fixed boundary conditions, e.g. enforcing null deformation or null velocity on the string only, we model the string as being connected to dynamical systems, as is common practice in accurate DW synthesis. The exceptional points are modeled by means of junctions

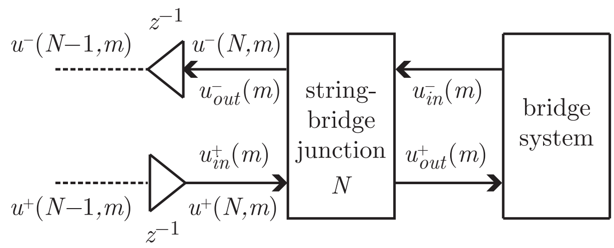

Figure 4: *Diagram of the digital waveguide connected to the bridge via a two-port junction.*

consisting of two-port elements (two inputs - two outputs). These elements characterize the change in the propagating waves due to the interaction occurring at the special points.

One of these systems represents the bridge and is located at $x = L$, where L is the length of the active portion of the string spanning from the nut (or fret if a finger is pressing the string on the fretboard) and the bridge. For simplicity, the length of the string L is assumed to be an integer multiple N of the spatial sampling interval: $L = NX$. Tuning the discrete model of the string also requires the use of additional fractional delays, which we will disregard in these preliminary considerations.

Another system represents the nut or the finger-fret coupling and is located at $x = 0$. Both nut/fret and bridge two-ports are terminated and the characteristics of the terminating element will ultimately influence the frequency dependent decay of the string sound. Terminations are implemented as linear time-invariant filters.

The third junction represents the plucking interaction for which a model was presented in [3], which allows for modeling the player's touch in the string excitation by means of a damped spring-mass system subject to external force.

Since the two systems modeling the bridge and the nut/finger are structurally similar, we will detail only the one representing the bridge in the following sections.

2.2. The Bridge Junction

At the bridge termination, a short but finite segment of the string is in contact with the bridge. All the static forces are assumed to balance each other and do not intervene in our discussion which concerns displacement with respect to the equilibrium position of the string. The interaction of the string with the bridge is modeled by the two-port block shown in Fig. 4, where the bridge system specifies the bridge termination of Fig. 3.

The incident traveling wave from the string continues to "travel" in the bridge and is partly reflected back into the string, possibly altered by multipath delay and frequency dependent damping. The way in which the vertical displacement of the string travels into the bridge can be described by the longitudinal compression of a bar, but it should also include all the other elastic elements in the chain from bridge to mounting screws and support. Since the string is usually snapped into a small notch, for the horizontal displacement, i.e. parallel to the soundboard, bending vibrations of the bridge can arise.

A similar type of mechanical interaction as that occurring at the bridge is also valid for modeling the response of the string at the nut and for modeling the imperfect finger-fret clamping of the string. The important point is that the model of the interaction of the string with the bridge or other element can be split into two components: a generic scattering element and the equivalent transfer function of the termination.

In the velocity wave representation of the solution for an analog waveguide, a model of the bridge termination can be given in terms of forces, velocities and driving-point impedances. In a simple model, the bridge and the two portions of the string share the same velocity:

$$V_b(s) = V_a(s) = V_p(s) \tag{3}$$

where $V_b(s)$, $V_a(s)$ and $V_p(s)$ respectively are the Laplace transforms of the velocities of the bridge, the active portion and the passive portion of the string at the string-bridge junction. The constant wave impedance $R_0 = \sqrt{K_0\mu}$ links both $V_a(s)$ and $V_p(s)$ to the corresponding force $F_a = R_0 V_a(s)$ and $F_p = R_0 V_p(s)$, while the complex, frequency dependent bridge impedance $R_b(s)$ links the bridge force F_b to bridge velocity: $F_b = R_b(s)V_b(s)$. Using the fact that the passive portion of the string is heavily damped at the tailpiece termination, one can assume that the corresponding velocity component $V_p(s)^+$ incident on the bridge is 0. In other words, one can treat the passive string segment as a transmission line terminated on its characteristic impedance.

Enforcing the equilibrium equation (the sum of the forces at the junction is 0) and using the methods in [8], one can easily derive the following reflectance, i.e. the transfer function for the overall bridge junction from V_a^+ to V_a^-:

$$H_b(s) = \frac{V_a^-(s)}{V_a^+(s)} = \frac{-1}{1 + 2R_0/R_b(s)}, \tag{4}$$

where the convention is that the $+$ superscript denotes velocity waves incident on the bridge junction and the $-$ superscript denotes velocity waves leaving the bridge junction.

In order to guarantee stability to the analog waveguide modeling the active portion of the string one must require that $H_b(s)$ is the transfer function of a passive system, i.e.

$$|H_b(j\omega)| \leqslant 1. \tag{5}$$

It can be shown that this is equivalent to requiring that $R_b(s)$ is a positive-real function, i.e. $R_b(s)$ is real for s real and $\Re\left[R_b(s)\right] \geqslant 0$ whenever $s > 0$, where $\Re$ denotes real part.

In order to adapt the analog description to a DW one must map $H_b(s)$ into the transfer function of a discrete-time filter, which can be achieved by means of the bilinear transform. In order to translate (4) into a form useful for displacement waves, one can multiply and divide both sides of the equation by s. In fact, in the Laplace transform, velocity wave variables are related to displacement variables as follows:

$$\frac{V^{\pm}(s)}{s} + C = U^{\pm}(s). \tag{6}$$

Let $u_a^-(t)$ and $u_b^+(t)$ respectively denote the inverse Laplace transforms of $U_b^+(s)$ and $U_a^-(s)$. In the diagram in Fig. 4 the quantities u_{out}^- and u_{in}^+ are discrete-time wave variables, respectively, corresponding to the displacement wave $u_a^-(t)$, leaving the bridge in the analog model, and to $u_b^+(t)$ entering the bridge. However, the additive constant C is arbitrary and the choice of the proper constant for the wave variables at the interface of two waveguides is non-trivial. In the next section we are going to introduce an intermediate "artificial" junction connecting the active portion of the string string to the bridge termination.

2.3. Multiplierless Scattering Junction

Suppose that the string and the bridge are each modeled by means of a different waveguide in the displacement wave representation. These two waveguides are connected at the string-bridge junction, whose behavior can be described by a constant 2×2 scattering matrix $\mathbf{S_b}$ linking the input to the output signals:

$$\begin{bmatrix} u^-_{out}(m) \\ u^+_{out}(m) \end{bmatrix} = \mathbf{S_b} \begin{bmatrix} u^-_{in}(m) \\ u^+_{in}(m) \end{bmatrix}. \quad (7)$$

The task of modeling the frequency dependent response of the string-bridge connection is delegated to the response of the waveguides. The inputs $u^-_{in}(m)$ and $u^+_{in}(m)$ to the junction are heterogeneous wave variables each pertaining to one waveguide and driven by the external force, velocity or displacement signals. Since the string and the bridge are in continuous contact with each other, the total displacements on each side of the junction must be identical. This gives us the condition:

$$u^-_{out}(m) + u^+_{in}(m) = u^-_{in}(m) + u^+_{out}(m). \quad (8)$$

Since it is immaterial if the string waveguide is on the left or on the right of the bridge junction, which happens, for example if the roles of the nut and the bridge are reversed, the scattering matrix should also be covariant by flipping. In other words, the changes $u^+_{in} \rightleftarrows u^-_{in}$ and $u^+_{out} \rightleftarrows u^-_{out}$ should leave the result unchanged. It is easy to see that (8) and the requirement that the matrix is symmetric are equivalent to this condition. The most general matrix satisfying these requirements has the form

$$\mathbf{S_b} = \begin{bmatrix} \alpha & \alpha - 1 \\ \alpha - 1 & \alpha \end{bmatrix}, \quad (9)$$

where α is an arbitrary constant. We remark that for $\alpha = 1$ we have the identity matrix, typical of a perfectly adapted connection without reflection (as for contiguous string segments with equal wave impedance). Moreover, for $\alpha = 0$ the junction corresponds to two completely disconnected waveguides, each reflecting waves into itself as a result of rigid termination. For $\alpha = 1/2$ we have

$$\mathbf{S_b} = \frac{1}{2} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}, \quad (10)$$

which provides a singular scattering matrix. In this case

$$\begin{aligned} u^-_{out}(m) &= \frac{1}{2} u^-_{in}(m) - \frac{1}{2} u^+_{in}(m) \\ u^+_{out}(m) &= -\frac{1}{2} u^-_{in}(m) + \frac{1}{2} u^+_{in}(m), \end{aligned} \quad (11)$$

and the equation

$$u^-_{out}(m) = -u^+_{out}(m) \quad (12)$$

is satisfied. This is an important property which says that the displacement wave signal transmitted to the bridge equals that reflected into the string with the sign reversed. This allows for variations of the displacement due to contact to originate from two signals adding to zero at the junction and traveling along the two waveguides.

To each of the wave signals one could add an arbitrary constant without changing the total displacement at any point, provided that an equal amount is subtracted from the other wave variable traveling on the opposite rail of each waveguide. The additive constants could differ in each waveguide. However, by virtue of (12), any constant mismatch is transmitted to the opposite rail of the other waveguide. Notice that (11) can be rewritten as follows:

$$\begin{aligned} u^-_{out}(m) + u^+_{in}(m) &= \frac{u^-_{in}(m) + u^+_{in}(m)}{2} \\ u^+_{out}(m) + u^-_{in}(m) &= \frac{u^-_{in}(m) + u^+_{in}(m)}{2}, \end{aligned} \quad (13)$$

which shows that the total displacements on both sides of the junction equal the average of the displacement waves incident on the junction from the string and the bridge reaction.

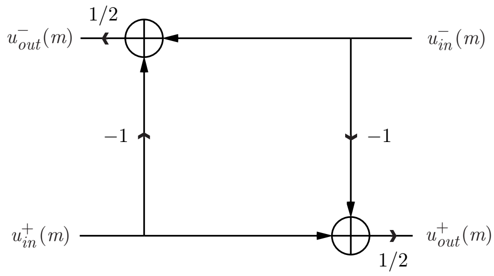

Figure 5: *Diagram of the scattering junction for the string-bridge connection.*

The diagram of the scattering junction connecting the active portion of the string with the bridge is shown in Fig. 5. The junction is multiplierless. Moreover, the four multiplications by $1/2$ (implementable as right-shift and complement operation in finite arithmetic) can be factored out to the two outputs.

Notice that with open right terminals (u^+_{out} and u^-_{in} not connected) only one half of the incoming wave u^+_{in} is reflected toward u^-_{out}. However, also notice that if $u^-_{in} = -u^+_{out}$ then $u^-_{out} = -u^+_{in}$ and perfect reflection is realized when the bridge is a rigid termination.

2.4. Transfer Function

Since the scattering matrix of the bridge junction is constant, the z-transform version of (7) can be written as follows:

$$\begin{bmatrix} U^-_{out}(z) \\ U^+_{out}(z) \end{bmatrix} = \mathbf{S_b} \begin{bmatrix} U^-_{\text{in}}(z) \\ U^+_{\text{in}}(z) \end{bmatrix}. \quad (14)$$

As shown in Fig. 3 the bridge acts as a termination to the bridge junction, whose behavior can be specified by a transfer function $G(z)$ relating U^+_{out} to U^-_{in}. In order to compute the overall string-bridge contact transfer function from U^+_{in} to U^-_{out}, one needs to determine the matrix $\hat{\mathbf{S}}_\mathbf{b}$ such that

$$\begin{bmatrix} U^-_{out}(z) \\ U^+_{\text{in}}(z) \end{bmatrix} = \hat{\mathbf{S}}_\mathbf{b} \begin{bmatrix} U^-_{\text{in}}(z) \\ U^+_{out}(z) \end{bmatrix}. \quad (15)$$

The matrix $\hat{\mathbf{S}}_\mathbf{b}$ is readily found by solving (14) for U^+_{in} and U^-_{out}, yielding

$$\hat{\mathbf{S}}_\mathbf{b} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}. \quad (16)$$

Hence,

$$\begin{bmatrix} U^-_{out}(z) \\ U^+_{\text{in}}(z) \end{bmatrix} = \hat{\mathbf{S}}_\mathbf{b} \begin{bmatrix} U^-_{\text{in}}(z) \\ U^+_{out}(z) \end{bmatrix} = \hat{\mathbf{S}}_\mathbf{b} \begin{bmatrix} G(z) \\ 1 \end{bmatrix} U^+_{\text{out}}(z). \quad (17)$$

Thus, the displacement wave reflectance, i.e. the overall transfer function $H_b(z)$ of the string-bridge connection from $U_{in}^+(z)$ to U_{out}^- is

$$H_b(z) = \frac{U_{out}^-(z)}{U_{in}^+(z)} = \frac{-1}{2+G(z)}. \tag{18}$$

The string-bridge interaction is fully specified once the termination transfer function $G(z)$ is specified.

The transfer function $H_b(z)$ is very similar to that derived in (4) for the analog velocity wave case using physical arguments. The main difference is that $H_b(z)$ is passive provided that $G(z)$ is passive, i.e., if $\left|G(e^{j\omega})\right| \leqslant 1$, while $H_b(s)$ is passive if $R_b(s)$ is positive real.

It must be pointed out that when the bridge is actually simulated by means of the given scattering plus termination, the termination transfer function $G(z)$ must either contain a delay or delay-free loops must be carefully handled as discussed in [9, 10]. Clearly, the process can be simplified by just implementing the transfer function (18) rather than the scattering matrix plus termination but, in this case, the system becomes less modular.

In turn, the transfer function $G(z)$ can be obtained by a digital waveguide model for the bridge. For example, one can consider a model inspired by the transverse, longitudinal or torsional vibrations of a bar, with different coupling modalities for the vertical and horizontal polarizations modes. In the transverse vibration case, dispersive behavior is observed where propagation delay is frequency dependent. Thus, in first approximation, $G(z)$ can be given as follows:

$$G(z) = -dA(z) \tag{19}$$

where $0 \leq d \leq 1$ is an overall damping factor and $A(z)$ is a suitable allpass filter modeling dispersive propagation. Moreover, the damping factor can be made frequency dependent, according to the reflectance of the bridge model at the bridge-body contact termination.

Alternately, the transfer function $G(z)$ can be estimated from measures of the bridge admittance $\Gamma(j\omega)$, which allows us to model a specific guitar. In [11] a passive parallel filter structure is developed for the direct modeling of the positive-real admittance. In our case, at least two different alternatives are available to model $H_b(z)$ via $G(z)$. In a first approach one can write $G(z) = A(z)$ where $A(z)$ is a real allpass transfer function. The allpass is formed by a number of second order allpass sections, each containing two complex conjugated first order sections, whose poles can be optimized in order to match the reflectance, in a design similar to the one proposed in [12].

The discrete counterpart of the reflectance is given by the z-transform

$$S(z) = \frac{\Gamma_0 - \Gamma(z)}{\Gamma_0 + \Gamma(z)} \tag{20}$$

where $\Gamma(z)$ is obtained from $\Gamma(s)$ by either spectrum invariance or bilinear transform and $\Gamma_0 = 1/R_0 = 1/\sqrt{K_0\mu}$ is the (constant) admittance of the guitar string. In guitars the reflectance function is in itself passive and almost allpass, as shown in Fig. 6. The rationale is that the magnitude frequency response $|H_b(e^{j\omega})|$ is maximum and equal to 1 where $A(e^{j\omega}) = -1$, i.e. at phase π, and minimum and equal to $1/3$, i.e. roughly 9.5 dB down, where $A(e^{j\omega}) = 1$. Moreover, the radius of the pole of the allpass filter $A(z)$ influences the bandwidth of the peaks of $|H_b|$. This is illustrated in Figs. 7 and 8, where the magnitude frequency response of the filter $H_b(z)$ is shown for various values of the angle and radii of the complex conjugate poles of a second order allpass

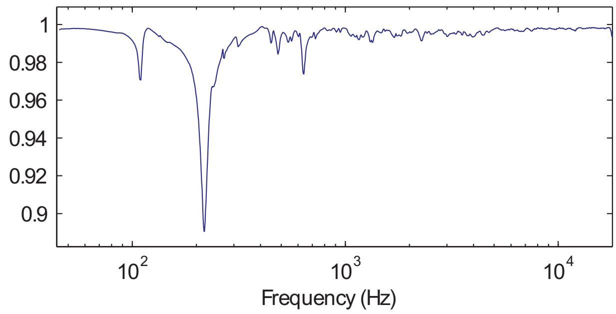

Figure 6: *Typical guitar bridge reflectance derived from admittance data.*

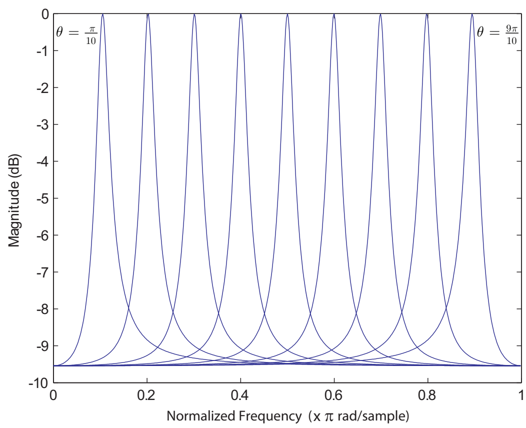

Figure 7: *Magnitude frequency response $H_b = -1/(2+A)$ for various angles θ of the poles of the second order allpass filter A (from $\theta = \pm\pi/10$ to $\theta = \pm 9\pi/10$), while keeping constant pole radius $\rho = 0.9$.*

filter. The results for angles different from $\pi/2$ are less symmetric, but are subject to similar interpretations. Moreover, the allpass is defined up to a leading sign. Changing the sign transforms the peaks in notches, useful in modeling reflectance curves.

In a higher order allpass design, the peaks at various frequency locations are superposed, as controlled by the angles and radius of the poles, as shown in the example of Fig. 9. The position of the poles can be optimized by using the Nelder-Mead simplex method with L_2 norm for the error (difference between synthetic and measured reflectance). However, in our experiments we observed a tendency to instability of this method as the number of allpass poles grows.

In an alternate and more stable design method, the transfer function $G(z)$ in (18) is expressed in terms of the reflectance derived from bridge admittance data as in (20):

$$G(z) = \frac{1}{S(z)} - 2 = \frac{3\Gamma - \Gamma_0}{\Gamma_0 - \Gamma}. \tag{21}$$

In this case, ARMA filter design can be achieved, which provides highly accurate minimum phase modeling of $G(z)$ and hence $H_b(z)$ from the admittance data, as shown in Fig. 10. While passivity is not guaranteed in this form, there is usually a margin (re-

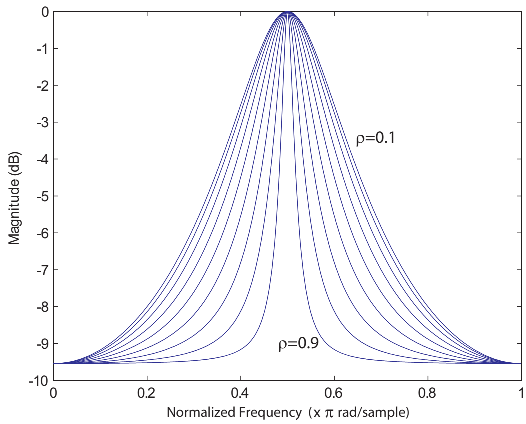

Figure 8: *Magnitude frequency response* $H_b = -1/(2 + A)$ *for various pole radii* ρ *of the poles of the second order allpass filter* A *(from* $\rho = 0.1$ *to* $\rho = 0.9$*), while keeping constant phase* $\pm\pi/2$.

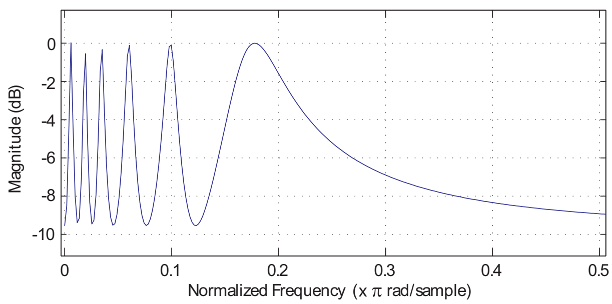

Figure 9: *Magnitude frequency response* $H_b = -1/(2 + A)$ *for order 12 allpass filter* A *with poles at various angles and radii.*

flectance curve slightly less than 1) so that the resulting filter turns out to be passive.

Besides the bridge model filter, the complete waveguide simulation also requires filters modeling frequency dependent air damping of the strings, which are typically lowpass.

In concluding this section, we remark that while the accurate model of the bridge admittance requires a higher order filter, the simple scheme modeling the reflectance as $H_b = -1/(2 - dA)$, where A is a low order allpass filter and d is a damping factor close to 1 provides quite realistic sounds at very low computational cost. Since the order 2 allpass filter maps twice the unit circle onto itself, the passbands of the filter are two in the notch filter configuration. In order to eliminate this effect, one can allow the damping factor d to be low-pass frequency dependent, which can be thought of as the loop filter of the waveguide simulating the bridge. Moreover, the damping factor d can be made frequency dependent and designed as a zero phase filter such that the overall frequency response matches the measured amplitudes of the peaks in the reflectance. This simplified design provides a good and low-cost alternative to the classical average filter in the Karplus-Strong scheme, which depends on a few adjustable parameters.

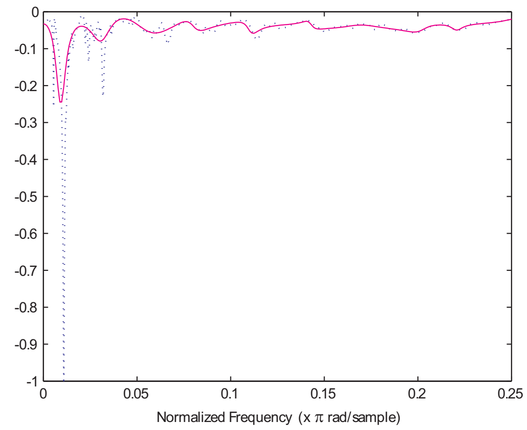

Figure 10: *Measured reflectance (dotted line) and magnitude frequency response of order 79 ARMA model (solid line) for bridge filter* H_b*, in the lower (1/4) frequency range.*

3. POLARIZATION AND SYMPATHETIC VIBRATIONS

Polarization of strings and the sympathetic vibrations of the other strings connected to the active string through the bridge are necessary ingredients for accurate guitar synthesis. These extensions we present are not novel (they can be found, e.g., in [2, 5, 6]), but they are implemented in the simplified bridge model in order to add to the realism of the generated sounds and to make the model complete. In the simplest case, the bridge is modeled by the following second order reflectance filter:

$$H_b = \frac{-1 + 2\Re b z^{-1} - |b|^2 z^{-2}}{2 - d|b|^2 - 2(2-d)\Re b z^{-1} + (2|b|^2 - d)z^{-2}} \quad (22)$$

where d is an overall damping factor, b is a complex number with $|b| < 1$ and $\Re b$ denotes the real part of b. Two filters, one for each polarization mode are implemented, which differ by the numerical values of the parameters b and d.

3.1. Polarization

The transverse vibrations of the string are two dimensional, contained in a plane orthogonal to the string's rest direction. In order to represent this, we decompose the motion into the vertical and horizontal components [7, 5, 6]. In the model we implemented, each string is represented by two waveguides of the type shown in Fig. 3. The two components interact with each other at exceptional points along the string.

First, at the bridge, the coupling of the two components implies transmission of energy between the horizontal and vertical components, making the string effectively rotate around its rest position. The angle θ is generally small and frequency dependent. A general coupling relation is developed in [7]:

$$\begin{bmatrix} u'_v \\ u'_h \end{bmatrix} = \begin{bmatrix} H_{vv}(z) & H_{vh}(z) \\ H_{hv}(z) & H_{hh}(z) \end{bmatrix} \begin{bmatrix} u_v \\ u_h \end{bmatrix} \quad (23)$$

where u'_v and u'_h respectively are the updated vertical and horizontal displacement waves at the bridge computed from u_v and u_h, the incoming vertical and horizontal displacement waves. $H_{vh}(z)$

and $H_{vh}(z)$ are cross-coupling and $H_{vv}(z)$ and $H_{hh}(z)$ are self-coupling transfer functions.

A simplified coupling relation is given by the rotation relation:

$$\begin{bmatrix} u'_v \\ u'_h \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} u_v \\ u_h \end{bmatrix} \tag{24}$$

which can be used in implementations with low computational cost, where the angle θ is independent of frequency.

Second, the bridge has a different effect on each component due to its material and geometry. As such, the filter applied will have different characteristics for each component. The horizontal polarization mode uses a different allpass in the reflectance filter $H_b(z)$ in (18) and (19) than that of the vertical polarization mode, corresponding to different decay rates. The different group delays of the reflectance filters also introduce a slight detuning of the two components which gives the guitar note its characteristic amplitude modulation.

In our experiments with the simplified second order reflectances (22), we found that reference values yielding acoustically satisfactory results for the pole position b are $b_v = 0.7e^{\pm j\frac{2\pi}{3}}$ for the vertical component bridge filter, and $b_h = 0.3e^{\pm j\frac{\pi}{5}}$ for the horizontal bridge filter.

Third, the attack angle is also taken into account at the plucking position. That means that the vertical projection of the attack stimulus is applied to the vertical component of the string, and identically the horizontal projection of the attack is applied to the horizontal component of the string.

3.2. Sympathetic vibrations

For even more realistic sound, sympathetic vibrations can be added to the model. Sympathetic vibrations are the vibration of the strings the player has no interaction with that are due to the transmission of energy through the bridge from the plucked string to the other strings.

In order to model this energy transfer, we use a coupling matrix as defined in [5, 13]. Considering that the gains of the transfered signals are symmetric, i.e. that the gain of the coupling from a to b is the same as from b to a, the bridge will couple N strings as follows:

$$U_v = C_{hv}U_h + H_{vv}U_v \tag{25}$$

and

$$U_h = C_{vh}U_v + H_{hh}U_h \tag{26}$$

where

$$U_v = (u_{v1}, u_{v2}, \dots, u_{vN})^T$$
$$U_h = (u_{h1}, u_{h2}, \dots, u_{hN})^T$$

and C_{hv} and C_{vh} are of the form

$$\begin{bmatrix} c_{11} & \dots & c_{1N} \\ \vdots & \ddots & \vdots \\ c_{N1} & \dots & c_{NN} \end{bmatrix} \tag{27}$$

The constants c_{km} correspond to the gain of the horizontal component of the kth string that is transmitted to the vertical component of the mth string.

An example of synthetically generated sympathetic vibrations is shown in Fig. 11, where the spectrogram of the vibration of the other simulated strings due to the played 5-th string is reproduced.

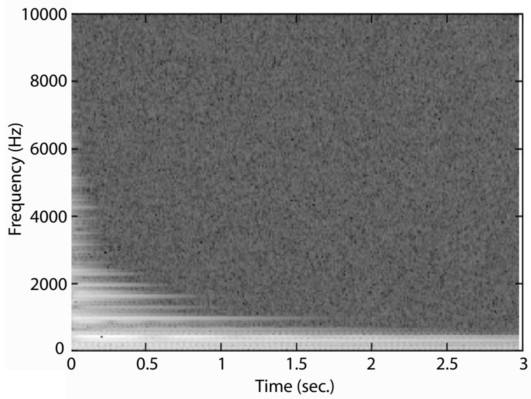

Figure 11: *Spectrogram of the sympathetic vibrations of other strings (waveguides) when plucking the synthetic open 5th string.*

4. RESULTS

The results produced with the use of the proposed bridge model are quite realistic even in the simple second order approximation of the bridge reflectance. The model we propose, by applying well chosen frequency dependent damping adds much naturalness to the sound. The frequency dependent decays of the partials can be changed by means of the poles of the allpass filter in (19) such as in (22) for example.

Figs. 12 and 13 show the spectrograms of 3 second sounds generated, respectively, with the Karplus-Strong model and with our model. We can see that the frequency-dependent decay in Fig. 13 is much closer to the one of a real guitar, in that higher frequencies disappear much quicker. This can be evaluated by means of the T30 decay time, which is the time in which a given frequency component decays by 30 dB with respect to the initial amplitude. Fig. 14 shows the T30 decay times of the first 10 harmonics of the synthetic sounds generated with different poles for the bridge transfer function (thin lines) compared to the T30 decay times of the 10 first harmonics of a real guitar note (thick line). Even if not identical, the T30 plots show that the synthetic and real curves follow the same trend. By implementing higher order filters (more poles) the decays of the synthetic harmonics can reach a good match with specific guitar sounds through parameter optimization.

Sound examples are available at:
http://staffwww.itn.liu.se/∼giaev/soundexamples.html.

5. CONCLUSION

Driven by the need for low computational cost, possibly scalable solutions, in this paper we provided a simplified model for the string-bridge interaction in the displacement wave variables. The model is part of the design of an evolving plugin for the synthesis of guitar introduced in [3]. The model of interaction is based on a simple multiplierless scattering junction (10), completed by a passive bridge termination transfer function $G(z)$, together forming the bridge reflectance filter (18). Extensions of the model include string polarization and sympathetic vibrations briefly discussed in Section 3. The model provides realistic sounds with simple design. Moreover, the scalable design of higher order filters matching prescribed or measured bridge termination characteristics and

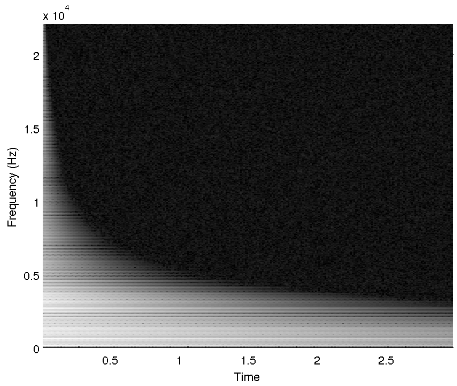

Figure 12: *Spectrogram of a guitar note generated with the Karplus-Strong bridge model.*

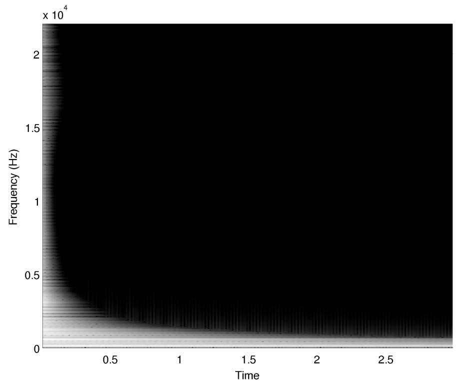

Figure 13: *Spectrogram of a guitar note generated with the presented bridge model.*

the modeling of the bridge transfer by means of a digital waveguide or waveguide mesh are possible.

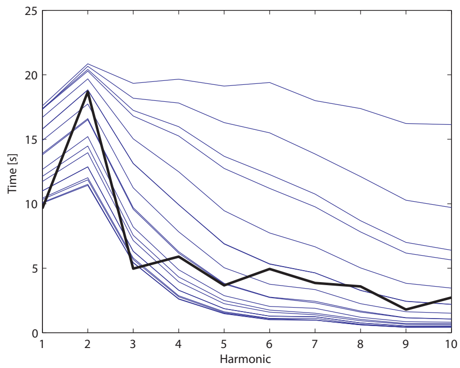

Figure 14: *T30 decay curves for the first 10 harmonics of the synthetic sounds generated with different magnitudes of the poles for the bridge transfer function (thin lines) compared to the T30 decay times of the 10 first harmonics of a real guitar note (thick line).*

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