PHYSICAL MODELLING OF A WAH-WAH EFFECT PEDAL AS A CASE STUDY FOR APPLICATION OF THE NODAL DK METHOD TO CIRCUITS WITH VARIABLE PARTS

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ABSTRACT

The nodal DK method is a systematic way to derive a non-linear state-space system as a physical model for an electrical circuit. Unfortunately, calculating the system coefficients requires inversion of a relatively large matrix. This becomes a problem when the system changes over time, requiring continuous recomputation of the coefficients. In this paper, we present an extension of the DK method to more efficiently handle variable circuit elements. The method is exemplified with the Dunlop Crybaby wah-wah effect pedal, as the continuous change of the potentiometer position is an extremely important aspect of the wah-wah effect.

1. INTRODUCTION

The nodal DK method is a systematic way to derive a non-linear state-space system as a physical model for an electrical circuit [1, 2]. Computation of the system coefficients involves inverting a system matrix, the size of which is determined by the number of circuit nodes, which is typically significantly higher than the order of the resulting system. While for time-invariant systems the inversion has to be performed only once, problems arise when variable elements like potentiometers are introduced. In that case, the computational load for continuous recalculation of the coefficients may become prohibitively large when following the straight-forward approach. Therefore, in this paper we present a more efficient recomputation scheme.

We exemplify the method with the Dunlop Crybaby wah-wah effect pedal, as the continuous change of the potentiometer position is an extremely important aspect of the wah-wah effect. Whether a full-featured non-linear physical model of the wah-wah is really necessary or simpler approaches (e.g. as described in [3]) suffice is anyone's choice; but the circuit is well suited to explain the technique presented in this paper which is why it is used here. The main part of the circuit is depicted in Figure 1. Omitted here are the input stage and the power-supply filter. The input stage is a simple AC-coupled emitter-follower which for reasonable input amplitudes may be well approximated by a linear high-pass filter. The output of that filter is used as the input voltage V_i. The supply voltage V_{cc} is obtained from the battery by an RC network which stabilizes the supply to about 90% of the battery voltage. The stabilization is sufficient to assume constant V_{cc}.

2. REVIEW OF THE DK METHOD

We shall start with a review of the nodal DK method because compared to [1], we use a slightly different discretization scheme for inductors and we need to more rigorously define the involved steps in terms of matrix operations in order to derive the simplified update scheme.

2.1. Companion circuits

The first step is to replace the energy-storing elements (capacitors and inductors) with so called companion circuits, obtained from discretization. In particular, from the differential equations

$$i_C = C\frac{d}{dt}v_C \qquad\qquad v_L = L\frac{d}{dt}i_L \tag{1}$$

for capacitor and inductor, by applying the trapezoidal discretization scheme, we get the discrete-time approximations

$$\tfrac{1}{2}\big(i_C(n)+i_C(n-1)\big) = \frac{C}{T}\big(v_C(n)-v_C(n-1)\big) \tag{2}$$

$$\tfrac{1}{2}\big(v_L(n)+v_L(n-1)\big) = \frac{L}{T}\big(i_L(n)-i_L(n-1)\big) \tag{3}$$

where T denotes the sampling interval. Solving for the current time step's currents then yields

$$i_C(n) = \frac{2C}{T}\big(v_C(n)-v_C(n-1)\big) - i_C(n-1) \tag{4}$$

$$i_L(n) = \frac{T}{2L}\big(v_L(n)+v_L(n-1)\big) + i_L(n-1), \tag{5}$$

where both the voltages and the currents hold state information. We may, however, introduce canonical states $x_C(n)$ and $x_L(n)$ by substituting

$$i_C(n) = x_C(n) - \frac{2C}{T}v_C(n) \tag{6}$$

$$i_L(n) = -x_L(n) - \frac{T}{2L}v_L(n) \tag{7}$$

in Equations (4) and (5) to get

$$x_C(n) - \frac{2C}{T}v_C(n) = \frac{2C}{T}v_C(n) - x_C(n-1) \tag{8}$$

$$-x_L(n) - \frac{T}{2L}v_L(n) = \frac{T}{2L}v_L(n) - x_L(n-1), \tag{9}$$

leading to the final state update equations

$$x_C(n) = 2\frac{2C}{T}v_C(n) - x_C(n-1) \tag{10}$$

$$x_L(n) = -2\frac{T}{2L}v_L(n) + x_L(n-1). \tag{11}$$

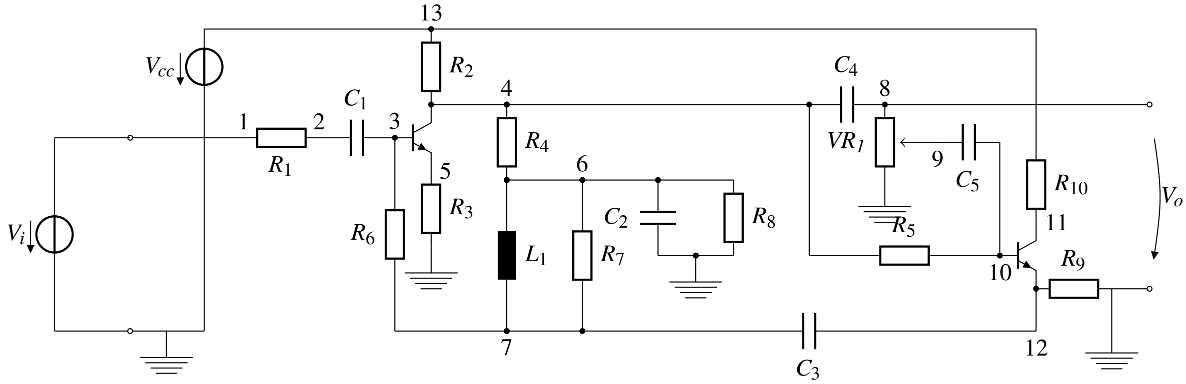

Figure 1: *Schematic of the analyzed Crybaby circuit with numbering of the nodes indicated.*

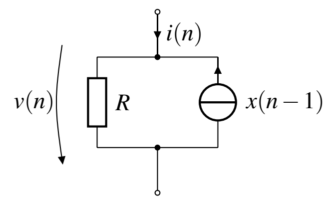

Figure 2: *Companion circuit for capacitor ($R = \frac{T}{2C}$) and inductor ($R = \frac{2L}{T}$).*

By substituting Equations (10) and (11) in Equations (6) and (7), we obtain

$$i_C(n) = \frac{2C}{T} v_C(n) - x_C(n-1) \tag{12}$$

$$i_L(n) = \frac{T}{2L} v_L(n) - x_L(n-1), \tag{13}$$

defining our companion circuit as shown in Figure 2. We hence replace all energy-storing elements with resistors and current sources, where the current sources depend on the previous time step and thereby hold the state information. We can thus analyze a circuit containing only resistors and sources (as described next) and then use the results to update the states (by Equations (10) and (11)), resulting in the next steps source currents.

2.2. Nodal K-method

The circuit is now analyzed using the nodal K-method. We define a reference node, the ground node as is common, and introduce the potentials φ_m of the other nodes and the currents $i_{s,i}$ through the voltage sources as unknowns. We now apply the Kirchhoff current law at all nodes except for the reference node. For example, for the circuit of Figure 1, the first three nodes yield the equations

$$\frac{1}{R_1}(\varphi_1 - \varphi_2) + i_{s,1} = 0 \tag{14}$$

$$\frac{1}{R_1}(\varphi_2 - \varphi_1) + \frac{1}{R_{C_1}}(\varphi_2 - \varphi_3) = x_{C_1} \tag{15}$$

$$\frac{1}{R_6}(\varphi_3 - \varphi_7) + \frac{1}{R_{C_1}}(\varphi_3 - \varphi_2) = -x_{C_1} - i_{B1} \tag{16}$$

where the current source of the companion circuit of C_1 is assumed to be pointing to the left. The transistor currents (the base current i_{B1} in the example equations) are for the moment assumed to be known and introduced as additional current sources.

The complete system may be written as

$$\begin{aligned}(\boldsymbol{N}_R^T \boldsymbol{G}_R \boldsymbol{N}_R + \boldsymbol{N}_v^T \boldsymbol{R}_v^{-1} \boldsymbol{N}_v + \boldsymbol{N}_x^T \boldsymbol{G}_x \boldsymbol{N}_x)\boldsymbol{\varphi} + \boldsymbol{N}_u^T \boldsymbol{i_s} \\ = \boldsymbol{N}_x^T \boldsymbol{x} + \boldsymbol{N}_n^T \boldsymbol{i}_n\end{aligned} \tag{17}$$

where $\boldsymbol{N}_R$, $\boldsymbol{N}_v$, $\boldsymbol{N}_x$, $\boldsymbol{N}_u$ and $\boldsymbol{N}_n$ are oriented incidence matrices which specify the nodes to which the resistors, potentiometers, energy-storing elements, voltage sources and non-linear elements, respectively, are connected,

$$\boldsymbol{G}_R = \text{diag}\left(\frac{1}{R_1}, \ldots, \frac{1}{R_{10}}\right) \tag{18}$$

is a diagonal matrix with the reciprocal resistances,

$$\boldsymbol{R}_v = \text{diag}\left(\alpha VR_1, (1-\alpha)VR_1\right) \tag{19}$$

is a diagonal matrix with the variable resistances parameterized with the potentiometer position α,

$$\boldsymbol{G}_x = \text{diag}\left(\frac{2C_1}{T}, \ldots, \frac{2C_5}{T}, \frac{T}{2L_1}\right) \tag{20}$$

is a diagonal matrix with the reciprocal resistances of the companion circuits, $\boldsymbol{\varphi}$ the vector of unknown node potentials, $\boldsymbol{i}_s$ the vector of unknown voltage source currents, $\boldsymbol{x}$ the current states of the companion circuits and $\boldsymbol{i}_n$ the currents of the non-linear elements.

The incidence matrices $\boldsymbol{N}_{(\cdot)}$ contain one row per circuit element and one column per node (except the reference node). The entries are mostly zero; at most two entries per element are nonzero: a 1 in the column of the node where the positive pole of the element is connected, a -1 where the negative pole is connected; connections to the reference node are omitted. Polarity of the sources is obvious, polarity of the passive elements may be chosen

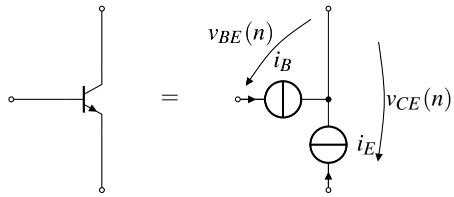

Figure 3: *Equivalent circuit element used for the transistors.*

at will. For the system of Figure 1, we find

$$\boldsymbol{N}_R = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \tag{21}$$

$$\boldsymbol{N}_v = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

$$\boldsymbol{N}_x = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{23}$$

$$\boldsymbol{N}_u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{24}$$

$$\boldsymbol{N}_n = \begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} \tag{25}$$

where we describe the transistors in terms of the base and emitter currents as shown in Figure 3.

To fully describe the circuit, we furthermore need to exploit that the voltage sources directly give relations among the node potentials, in our case

$$\varphi_1 = V_i \tag{26}$$
$$\varphi_{13} = V_{cc} \tag{27}$$

which may be combined with Equation (17) to give

$$\boldsymbol{S}\begin{pmatrix}\boldsymbol{\varphi} \\ \boldsymbol{i_s}\end{pmatrix} = \begin{pmatrix}\boldsymbol{N}_x^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{I}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}\boldsymbol{N}_n^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{i}_n \tag{28}$$

where $\boldsymbol{u} = \begin{pmatrix}V_i & V_{cc}\end{pmatrix}^T$ is the source vector, $\boldsymbol{0}$ and $\boldsymbol{I}$ are the all-zero and identity matrix with size as required by context, and

$$\boldsymbol{S} = \begin{pmatrix}\boldsymbol{N}_R^T\boldsymbol{G}_R\boldsymbol{N}_R + \boldsymbol{N}_v^T\boldsymbol{R}_v^{-1}\boldsymbol{N}_v + \boldsymbol{N}_x^T\boldsymbol{G}_x\boldsymbol{N}_x & \boldsymbol{N}_u^T \\ \boldsymbol{N}_u & \boldsymbol{0}\end{pmatrix}. \tag{29}$$

By left-multiplying with $\boldsymbol{S}^{-1}$ and the respective incidence matrices, we can then extract all required voltages by

$$\boldsymbol{v}_x = \begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\left(\begin{pmatrix}\boldsymbol{N}_x^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{I}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}\boldsymbol{N}_n^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{i}_n\right) \tag{30}$$

$$\boldsymbol{v}_n = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\left(\begin{pmatrix}\boldsymbol{N}_x^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{I}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}\boldsymbol{N}_n^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{i}_n\right) \tag{31}$$

$$\boldsymbol{v}_o = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\left(\begin{pmatrix}\boldsymbol{N}_x^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{x} + \begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{I}\end{pmatrix}\boldsymbol{u} + \begin{pmatrix}\boldsymbol{N}_n^T \\ \boldsymbol{0}\end{pmatrix}\boldsymbol{i}_n\right) \tag{32}$$

where $\boldsymbol{v}_x$, $\boldsymbol{v}_n$ and $\boldsymbol{v}_o$ are the voltages across the energy-storing elements, the non-linear elements and the desired output voltage, respectively. The incidence matrix $\boldsymbol{N}_o$ determines the output voltage and in the example is given by

$$\boldsymbol{N}_o = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

2.3. Non-linear state-space system

We are now ready to formulate the non-linear state space system. We first rewrite Equations (10) and (11) for a complete system as

$$\boldsymbol{x}(n) = \boldsymbol{Z}\cdot\left(2\boldsymbol{G}_x\boldsymbol{v}_x(n) - \boldsymbol{x}(n-1)\right) \tag{34}$$

where

$$\boldsymbol{Z} = \operatorname{diag}\begin{pmatrix}1 & 1 & 1 & 1 & 1 & -1\end{pmatrix} \tag{35}$$

is a diagonal matrix which contains a 1 for a capacitor and a -1 for an inductor to take care of the differing signs in Equation (10) and Equation (11). Substituting Equation (30), we obtain the state update equation

$$\boldsymbol{x}(n) = \boldsymbol{A}\boldsymbol{x}(n-1) + \boldsymbol{B}\boldsymbol{u}(n) + \boldsymbol{C}\boldsymbol{i}_n(n) \tag{36}$$

with

$$\boldsymbol{A} = 2\boldsymbol{Z}\boldsymbol{G}_x\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}^T - \boldsymbol{Z} \tag{37}$$
$$\boldsymbol{B} = 2\boldsymbol{Z}\boldsymbol{G}_x\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}^T \tag{38}$$
$$\boldsymbol{C} = 2\boldsymbol{Z}\boldsymbol{G}_x\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}^T. \tag{39}$$

Similarly, we may rewrite Equation (32) to obtain the output equation

$$\boldsymbol{y}(n) = \boldsymbol{v}_o(n) = \boldsymbol{D}\boldsymbol{x}(n-1) + \boldsymbol{E}\boldsymbol{u}(n) + \boldsymbol{F}\boldsymbol{i}_n(n) \tag{40}$$

with

$$\boldsymbol{D} = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}^T \tag{41}$$
$$\boldsymbol{E} = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}^T \tag{42}$$
$$\boldsymbol{F} = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}^T. \tag{43}$$

Finally, we obtain the voltages across the non-linear elements by rewriting Equation (31) to

$$\boldsymbol{v}_n(n) = \boldsymbol{G}\boldsymbol{x}(n-1) + \boldsymbol{H}\boldsymbol{u}(n) + \boldsymbol{K}\boldsymbol{i}_n(n) \tag{44}$$

with

$$\boldsymbol{G} = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}^T \tag{45}$$
$$\boldsymbol{H} = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}^T \tag{46}$$
$$\boldsymbol{K} = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}^{-1}\begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}^T. \tag{47}$$

Note that Equation (44) defines a relationship between the voltages across and the currents through the non-linear circuit elements due to the external circuit and its state. Additionally, the non-linear elements define such a relationship by themselves; combining these two will allow to solve for the currents $\boldsymbol{i}_n(n)$ or equivalently the voltages $\boldsymbol{v}_n(n)$.

In the example, the non-linear elements are transistors which will be modelled using the Ebers-Moll-equations. In particular, we get

$$i_{n,1} = \frac{I_s}{\beta_F}\left(e^{\frac{v_{n,2}-v_{n,1}}{V_t}}-1\right)+\frac{I_s}{\beta_R}\left(e^{\frac{-v_{n,1}}{V_t}}-1\right) \tag{48}$$

$$i_{n,2} = -I_s\left(e^{\frac{v_{n,2}-v_{n,1}}{V_t}}-1\right)+\frac{I_s(\beta_R-1)}{\beta_R}\left(e^{\frac{-v_{n,1}}{V_t}}-1\right) \tag{49}$$

$$i_{n,3} = \frac{I_s}{\beta_F}\left(e^{\frac{v_{n,4}-v_{n,3}}{V_t}}-1\right)+\frac{I_s}{\beta_R}\left(e^{\frac{-v_{n,3}}{V_t}}-1\right) \tag{50}$$

$$i_{n,4} = -I_s\left(e^{\frac{v_{n,4}-v_{n,3}}{V_t}}-1\right)+\frac{I_s(\beta_R-1)}{\beta_R}\left(e^{\frac{-v_{n,3}}{V_t}}-1\right). \tag{51}$$

The processing for time step n now proceeds according to the following schedule:

1. Calculate $\boldsymbol{p}(n) = \boldsymbol{G}\boldsymbol{x}(n-1)+\boldsymbol{H}\boldsymbol{u}(n)$.
2. Numerically solve $\boldsymbol{p}(n)+\boldsymbol{K}\boldsymbol{i}_n(n)-\boldsymbol{v}_n(n)=0$ together with Equations (48) to (51) to obtain $\boldsymbol{i}_n(n)$, e.g. using Newton iteration.
3. Compute the output with Equation (40).
4. Perform the state update with Equation (36).

3. EFFICIENT HANDLING OF VARIABLE ELEMENTS

As all of the matrices defined above can be precomputed, the main computational burden lies in solving the non-linear equation. However, in case of a variable element, like the potentiometer *VR1*, the system matrices need to be recomputed every time the element changes—and for the wah-wah effect, the potentiometer will change almost continuously. At first glance, it looks like this means the matrix $\boldsymbol{S}$ has to be inverted regularly, which would be rather unfortunate, as inverting a 15×15 matrix would mean a significant additional computational load.

Fortunately, we may decompose the system matrix as

$$\boldsymbol{S} = \boldsymbol{S}_0 + \begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T \boldsymbol{R}_v^{-1} \begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix} \tag{52}$$

where

$$\boldsymbol{S}_0 = \begin{pmatrix}\boldsymbol{N}_R^T\boldsymbol{G}_R\boldsymbol{N}_R+\boldsymbol{N}_x^T\boldsymbol{G}_x\boldsymbol{N}_x & \boldsymbol{N}_u^T \\ \boldsymbol{N}_u & \boldsymbol{0}\end{pmatrix} \tag{53}$$

is independent of the potentiometer setting. We may now use the Woodbury identity [4] to rewrite the inverse as

$$\boldsymbol{S}^{-1} = \boldsymbol{S}_0^{-1} - \boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1} \tag{54}$$

with

$$\boldsymbol{Q} = \begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T \tag{55}$$

which reduces the size of the matrix to be regularly inverted to 2×2, as $\boldsymbol{S}_0^{-1}$ may be precomputed off-line. As an additional benefit, we do not have to worry about inverting $\boldsymbol{R}_v$, which would require special attention for $\alpha=0$ and $\alpha=1$ when the resistances become zero. Plugging Equation (54) into the definition of the state-space system matrices, we obtain

$$\boldsymbol{A} = \boldsymbol{A}_0 - 2\boldsymbol{Z}\boldsymbol{G}_x\boldsymbol{U}_x(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_x^T \tag{56}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 - 2\boldsymbol{Z}\boldsymbol{G}_x\boldsymbol{U}_x(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_u^T \tag{57}$$

$$\boldsymbol{C} = \boldsymbol{C}_0 - 2\boldsymbol{Z}\boldsymbol{G}_x\boldsymbol{U}_x(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_n^T \tag{58}$$

$$\boldsymbol{D} = \boldsymbol{D}_0 - \boldsymbol{U}_o(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_x^T \tag{59}$$

$$\boldsymbol{E} = \boldsymbol{E}_0 - \boldsymbol{U}_o(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_u^T \tag{60}$$

$$\boldsymbol{F} = \boldsymbol{F}_0 - \boldsymbol{U}_o(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_n^T \tag{61}$$

$$\boldsymbol{G} = \boldsymbol{G}_0 - \boldsymbol{U}_n(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_x^T \tag{62}$$

$$\boldsymbol{H} = \boldsymbol{H}_0 - \boldsymbol{U}_n(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_u^T \tag{63}$$

$$\boldsymbol{K} = \boldsymbol{K}_0 - \boldsymbol{U}_n(\boldsymbol{R}_v+\boldsymbol{Q})^{-1}\boldsymbol{U}_n^T \tag{64}$$

with the constant matrices

$$\boldsymbol{U}_x = \begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T \tag{65}$$

$$\boldsymbol{U}_o = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T \tag{66}$$

$$\boldsymbol{U}_n = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T \tag{67}$$

$$\boldsymbol{U}_u = \begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_v & \boldsymbol{0}\end{pmatrix}^T \tag{68}$$

$$\boldsymbol{A}_0 = 2\boldsymbol{Z}\boldsymbol{G}_x\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}^T - \boldsymbol{Z} \tag{69}$$

$$\boldsymbol{B}_0 = 2\boldsymbol{Z}\boldsymbol{G}_x\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}^T \tag{70}$$

$$\boldsymbol{C}_0 = 2\boldsymbol{Z}\boldsymbol{G}_x\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}^T \tag{71}$$

$$\boldsymbol{D}_0 = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}^T \tag{72}$$

$$\boldsymbol{E}_0 = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}^T \tag{73}$$

$$\boldsymbol{F}_0 = \begin{pmatrix}\boldsymbol{N}_o & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}^T \tag{74}$$

$$\boldsymbol{G}_0 = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_x & \boldsymbol{0}\end{pmatrix}^T \tag{75}$$

$$\boldsymbol{H}_0 = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{0} & \boldsymbol{I}\end{pmatrix}^T \tag{76}$$

$$\boldsymbol{K}_0 = \begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}\boldsymbol{S}_0^{-1}\begin{pmatrix}\boldsymbol{N}_n & \boldsymbol{0}\end{pmatrix}^T. \tag{77}$$

Note that all matrices involved in Equations (56) to (64) are relatively small compared to the system matrix $\boldsymbol{S}$, allowing efficient recomputation whenever the potentiometer setting changes.

4. RESULTS

The above model, supplemented with a simple first-order high-pass for the AC-coupled input buffer omitted from the above circuit analysis, has been implemented as an LV2[1] plugin. The matrix operations make use of the GNU Scientific Library[2]. The non-linear equation is solved with a simple damped Newton iteration, where the solution of the previous time-step is used as starting point. This allows convergence in relatively few (typically less than ten) iterations, making real-time operation even on not quite up-to-date PC hardware possible. The plugin along with audio examples may be found at `http://ant.hsu-hh.de/dafx2011/wahwah`.

The model has been tested using the nominal component values and parameters given in Table 1. As a first test, the small-signal frequency responses of the model and the real circuit are compared

[1] See `http://lv2plug.in/`.
[2] See `http://www.gnu.org/software/gsl/`.

Table 1: *Component values and parameters used in the model.*

R_1	68	kΩ	C_1	10	nF
R_2	22	kΩ	C_2	4.7	µF
R_3	390	Ω	C_3	10	nF
R_4	470	kΩ	C_4	220	nF
R_5	470	kΩ	C_5	220	nF
R_6	1.5	kΩ	L_1	500	mH
R_7	33	kΩ	V_{cc}	8.15	V
R_8	82	kΩ	V_t	26	mV
R_9	10	kΩ	I_s	20.3	fA
R_{10}	1	kΩ	β_f	1430	
VR_1	100	kΩ	β_r	4	

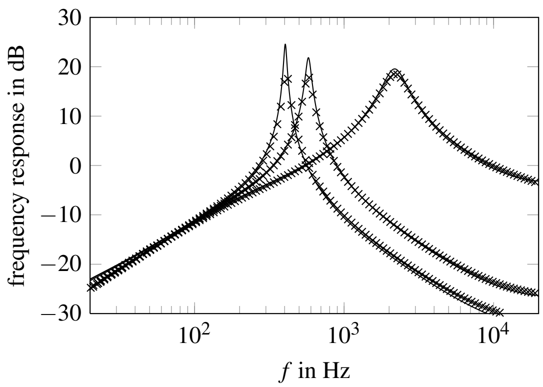

Figure 4: *Comparison of measured (crosses) and simulated (line) small-signal frequency responses for* $\alpha = 0.008, 0.570,$ *and* 0.999.

for three different potentiometer settings[3]. The corresponding value of α was determined by measuring the potentiometer. The resulting frequency responses are shown in Figure 4. Clearly, the model fits the real circuit quite well. Any deviations are easily explained by the tolerances of the physical circuit elements; especially the slightly different gain at the peak is no surprise, as the relative high Q-factor makes the circuit very sensitive to component tolerances.

The second experiment conducted is concerned with the non-linear behavior. Even for moderate input levels, the high gain at the peak frequency may result in a clipped output signal. As a very simple form of analysis, Figure 5 depicts the output level and total harmonic distortion (THD) in dependence on the input level for a sinusoidal input. The frequency of the sinusoid is chosen to be at the peak of the small-signal frequency response. As expected, for small input amplitudes, the simulated output has slightly higher output level corresponding to the higher small-signal gain at the peak frequency. For higher input amplitudes, both the real circuit and the simulation exhibit a smooth saturation when the output level approaches 6 dB. This smooth saturation is also resembled by the THD curve, which shows a steady increase for both measurement and simulation and stays at moderate levels even for a 0 dB input, where the overall gain is reduced by about 13 dB compared to the small-signal gain. While the THD of the simulation is constantly a few dB higher than that of the real circuit, the shape of the curves is very similar and it may be expected that better matching of the component values and the transistor parameters would bring the curves to close fit.

[3] Note that the extremal positions cannot be reached due to mechanical restrictions when the potentiometer is mounted in the enclosure.

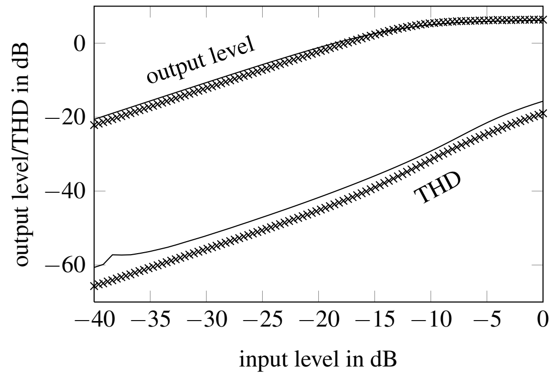

Figure 5: *Comparison of measured (crosses) and simulated (line) output RMS level and THD in dependence on input RMS level for a sinusoid of* 719 Hz *with potentiometer setting* $\alpha = 0.770$.

5. CONCLUSION

We have reviewed the nodal DK-method with a focus on efficient handling of variable circuit components, i.e. potentiometers. To that end, we have given a formulation of the DK-method with explicit use of the incidence matrices and diagonal matrices holding the component values, from which a system matrix is constructed that has to be inverted to obtain the coefficient matrices of a non-linear state-space model. This explicit form allows efficient handling of changing component values when the number of variable parts is low compared to the total number of parts. In that case, changing the variable component values results in a low-rank update of the system matrix, and the required inversion may be carried out efficiently using the Woodbury identity.

The method was presented by example of the Crybaby wah-wah circuit. The derived model provides a reasonable approximation of the real circuit considering that the nominal component values where used and no parameter matching to the real circuit was performed. The model was implemented as an LV2 plugin which proved the real-time capability of the model even on somewhat outdated PC hardware.

6. REFERENCES

[1] D.T. Yeh, J.S. Abel, and J.O. Smith, "Automated physical modeling of nonlinear audio circuits for real-time audio effects—Part I: Theoretical development," *IEEE Trans. Audio, Speech, and Language Process.*, vol. 18, no. 4, pp. 728–737, May 2010.

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