LP-BLIT: BANDLIMITED IMPULSE TRAIN SYNTHESIS OF LOWPASS-FILTERED WAVEFORMS

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ABSTRACT

Using bandlimited impulse train (BLIT) synthesis, it is possible to generate waveforms with a configurable number of harmonics with an equal amplitude. In contrast to the sinc-pulse, which is typically used for bandlimiting in BLIT and only allows to set the cutoff frequency, a Hammerich pulse can be tuned by two independent parameters for cutoff frequency and stop band roll-off. Replacing the perfect lowpass sinc-pulse in BLIT with a Hammerich pulse, it is possible to directly synthesise a multitude of signals with an adjustable lowpass spectrum.

1. INTRODUCTION

Subtractive sound synthesis with analogue synthesisers requires generic and spectrally rich harmonic waveforms. Traditionally, these are rectangular, triangular and sawtooth waveforms. Their flat spectrum is successively shaped by filters until it matches the expectations of the musician. These filters usually have a low-pass characteristic with a variable cutoff frequency, adjustable resonance peak and a slope of 12 or 24 dB per octave. One major problem of digital subtractive synthesis is the fact that trivial implementations of the required basic waveforms leads to massive aliasing. The creation of bandlimited oscillators has been an active research topic ever since digital sound synthesis became of interest. An extensive summary and discussion of various methods can be found in [1] and [2]. The latter also introduced the Poly-BLEP approach (detailed in [3]) which today is a standard method to create high quality bandlimited waveforms due to its low computational cost and easy implementation.

Subtractive synthesis is a straightforward approach when building analogue synthesisers. Today, many digital synthesisers mimic an analogue subtractive workflow, mainly due to the fact that musicians have been used to it for decades and the use of filters to shape the sound is at the same time intuitive, simple and versatile. However, for the performing musician the exact synthesis method does not matter. It is more important that the synthesiser allows to create musically sounding signals which can be intuitively controlled by only a few but powerful parameters [4]. From an engineering point of view it does not make sense to put a lot of effort into the generation of aliasing-free waveforms with harmonics up to half the sampling frequency and then to remove most of the high frequency content with a lowpass filter. A method to directly synthesise the desired signal spectra would alleviate the effort for anti-aliasing strategies as such a signal will contain less high frequency content from the beginning. Additive synthesis as a discrete summation of amplitude-weighted sine waves would offer full control to the creation process of perfectly bandlimited signals but due to the resulting computational complexity it is rarely used in practice.

The discrete summation formula (DSF) from Moorer [5] are a set of closed form solutions to discrete harmonic series (infinite or finite). They allow a direct synthesis of harmonic signals with a specified number of partials and an exponentially increasing or decreasing spectral envelope. By the combination of differently parametrised DSF one can create further complex spectral envelopes. Although the DSF are more efficient than a discrete additive synthesis they still require a considerable amount of trigonometric function evaluations.

Bandlimited impulse train (BLIT) synthesis [1] is another approach to create lowpass signals and relies on the fact that an impulse train exhibits a flat spectrum with an infinite amount of harmonics. When the impulse train is convolved with a sinc-function, a perfect bandlimited signal will be obtained, whereas the frequency of the sinc-pulse determines the cutoff frequency. In practical applications, the infinite length sinc-function has to be windowed and limited to a reasonable length and the computationally expensive convolution is replaced by a summation of overlapping sinc-pulses with a pulse distance proportional to the fundamental frequency (sum of windowed sinc-function BLIT synthesis [1]). A windowed sinc-function will loose its perfect lowpass characteristic and stop-band ripple occurs. The selection and parametrisation of the window and its length influences the final lowpass shape [6]. BLIT synthesis was further developed and optimised in various aspects (e.g. in [7, 8]), but the created signals have been always used as a bandlimited input for subtractive workflows. To the knowledge of the authors, a direct BLIT synthesis of signals with real-time configurable lowpass spectra has not been investigated so far.

The Hammerich pulse [9] was introduced as a pulse shape filter [10] for transmission systems and its spectral shape can be tuned by two independent parameters for cutoff frequency and stop band roll-off. Replacing the sinc-pulses in BLIT with Hammerich pulses allows to directly synthesise signals with adjustable low-pass spectra. The fundamental frequency, cutoff frequency and stop band roll-off are monotonic parameters and can be modulated smoothly. In a sound synthesis application, the few parameters and inherent limitation to lowpass spectra reduces complexity in the user interface and offers the musician an intuitive and less technical access. Nevertheless, a wide variety of sounds similar to subtractive synthesisers can be created without additional filtering. Moreover, unique sounds can be generated for example by modulating the filter roll-off which would not be possible with classical analogue synthesisers.

The idea to create a versatile oscillator with Hammerich pulse shapes was already briefly described in [11]. In this paper, the focus will be on a detailed discussion of such a lowpass bandlimited impulse train (LP-BLIT) oscillator in the context of sound synthesis applications. Throughout the following Section 2, the principle of BLIT synthesis and the integration of the Hammerich pulse will be explained. In Section 3 we will give some examples how to create more complex spectra by combination of several oscillator outputs, leaky integration and parameter modulation before Section 4 provides a short discussion and conclusion.

2. IMPULSE TRAIN SYNTHESIS

A continuous-time pulse train with an inter-pulse distance T_0

$$d(t) = \sum_{l=-\infty}^{\infty} \delta(t - l\, T_0) \tag{1}$$

exhibits a spectrum

$$D(j\omega) = \sum_{l=-\infty}^{\infty} \delta(\omega - l\, \omega_0), \qquad \omega_0 = \frac{2\,\pi}{T_0} \tag{2}$$

with harmonics at multiples of ω_0. The fact that the harmonic interval ω_0 is directly related to T_0 can be utilised to synthesise a signal with an infinite amount of harmonics and a fundamental frequency $F_0 = 1/T_0$. Sampling of $d(t)$ for digital implementations would lead to massive aliasing due to its infinite bandwidth. Applying an ideal lowpass filter with a cutoff frequency $\omega_c = 2\pi f_c$

$$h(t) = \frac{\sin(\omega_c\, t)}{\omega_c\, t} \quad \leftrightarrow \quad H(j\omega) = \mathrm{rect}\left(\frac{\omega}{2\,\omega_c}\right) \tag{3}$$

results in a bandlimited signal

$$x(t) = d(t) * h(t) \tag{4}$$

which can then be sampled at a sample rate $f_s > 2\, f_c$ without aliasing [1]. In the frequency domain, the convolution from Eq. 4 will create a harmonic spectrum

$$\begin{aligned} X(j\omega) &= H(j\omega) \cdot D(j\omega) \\ &= H(j\omega) \cdot \left(\sum_{l=-\infty}^{\infty} \delta(j\omega - l\, \omega_0) \right) \end{aligned} \tag{5}$$

weighted with the spectral shape H of the filter. By replacing or modifying the filter, it is possible to synthesise any bandlimited harmonic signal with an arbitrary spectral shape.

Instead of a computationally expensive time-domain convolution, a direct summation of time-shifted impulse responses

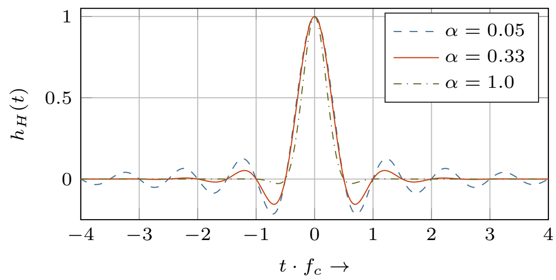
$$x(t) = d(t) * h(t) = \sum_{l=-\infty}^{\infty} h(t - l\, T_0) \tag{6}$$

will lead to an identical result. This is particularly useful if the impulse response can be simply calculated with a closed-form equation as it is the case with the sinc-function.

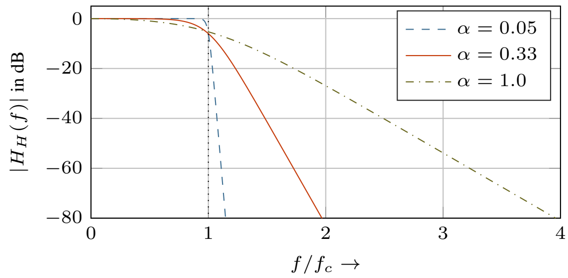
Other pulses with a lowpass characteristic, as for example the raised cosine pulse, offer a more detailed adjustment of their frequency response than the sinc-function. The Hammerich pulse was introduced in [10] as a pulse shape filter for transmission systems. Its impulse and frequency response is given by

$$h_H(t) = \frac{\alpha \cdot \sin(\omega_c\, t)}{\sinh(\alpha \cdot \omega_c\, t)} \tag{7}$$

$$H_H(j\omega) = \frac{1}{4 f_c} \left[1 - \tanh\left(\frac{|\omega| - \omega_c}{4 \alpha f_c} \right) \right] \tag{8}$$



(a) Time domain



(b) Frequency domain

Figure 1: *Hammerich pulse for various values of* α*.*

and permits an intuitive adjustment of the lowpass characteristic by two independent parameters for cutoff frequency ω_c and stop band slope α. Reasonable parameter ranges are $0 < \alpha < 10$ and $\omega_c > 2\pi F_0$. For small values of α, the impulse response

$$\lim_{\alpha \to 0} h_H(t) = \frac{\sin(\omega_c\, t)}{\omega_c\, t} = \mathrm{sinc}(\omega_c\, t) \tag{9}$$

converges towards a sinc-function and for $\alpha \to \infty$ the resulting stop band slope as well as the pulse width converges to zero. Exemplary impulse and frequency responses are depicted in Fig. 1 for selected values of α. The -6 dB point is fairly accurate set by ω_c, whereas the linear slope beyond this point is only controlled by α. Figure 2 depicts a time-domain pulse train after convolution with a Hammerich impulse response together with the corresponding spectrum. The parameters of the Hammerich filter were chosen as $f_c = 4\, F_0$ and $\alpha = 0.4$. All harmonics follow the shape of the pulse spectrum as it was expected based on Eq. 5.

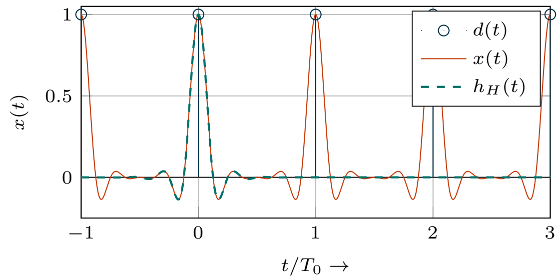
2.1. Discrete-time implementation with finite pulse length

By sampling the Hammerich pulse from Eq. 7 with a sample rate f_s, the discrete-time pulse

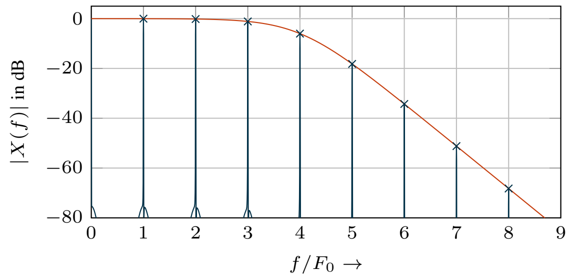
$$h_H(n) = \frac{\alpha \cdot \sin(\Omega_c\, n)}{\sinh(\alpha \cdot \Omega_c\, n)}, \qquad \Omega_c = 2\pi \frac{f_c}{f_s}, \tag{10}$$

is obtained. The cutoff frequency $f_c = N_h \cdot F_0$ can also be expressed as a multiple of the fundamental frequency, whereas $N_h \geq 1$ determines the number of harmonics before the filter starts to roll off. This yields a pulse

$$h_H(n) = \frac{\alpha \cdot \sin(N_h \cdot \Omega_0\, n)}{\sinh(\alpha \cdot N_h \cdot \Omega_0\, n)}, \qquad \Omega_0 = 2\pi \frac{F_0}{f_s}, \tag{11}$$

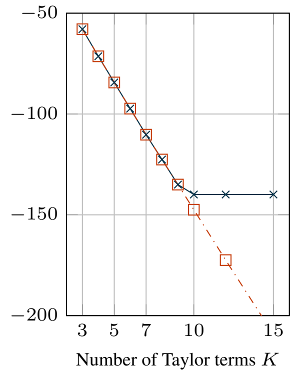


(a) *Time domain*

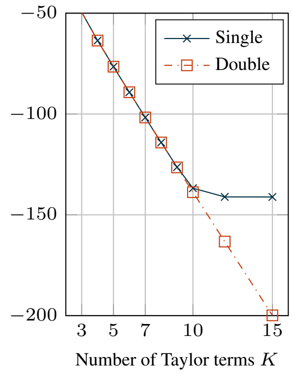


(b) *Frequency domain*

Figure 2: *Pulse train convolved with a Hammerich pulse ($f_c = 4\,F_0$ and $\alpha = 0.4$).*



(a) *Maximum error*



(b) *Power normalised MSE*

Figure 3: *Error after the approximation of $h_H(n)$ with K Taylor series terms for single and double precision floating point implementations.*

with parameters for the number of harmonics, filter slope and fundamental frequency being directly accessible from a synthesiser application.

Two facts have to be considered for a discrete-time implementation compared to the continuous-time derivation in the previous section. First, the theoretically infinite length of the pulse has to be limited in order to avoid an infinite amount of overlapping pulses in the sum from Eq. 6. This limitation of the impulse length is equivalent to a windowing of the impulse response and leads to a distortion of the pulse spectrum. The shape of the resulting error can be optimised by applying a smooth window to both ends of the pulse [6]. The number of overlapping pulses is a trade-off between computational complexity and how close the actual spectral envelope will match the theoretic spectrum given in Eq. 8. Second, even if the cutoff frequency of the Hammerich pulse is below $f_s/2$, aliasing may occur due to the flat spectral roll-off after f_c depending on the actual selection of α. Hence, it is necessary to limit the combination of the parameters N_h and α such that the resulting stop band attenuation in Eq. 8 falls below an acceptable level at half the sample rate.

Regarding computational complexity, the calculation of the sine and hyperbolic sine in the Hammerich pulse is the limiting factor. Both functions could be approximated by the Taylor series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{K-1} \frac{(-1)^k \cdot x^{2k+1}}{(2k+1)!} \tag{12}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots = \sum_{k=0}^{K-1} \frac{x^{2k+1}}{(2k+1)!} \tag{13}$$

where the number of evaluated terms K determines the accuracy. For the periodic sine, the argument has to be wrapped to a range $x \in [-\pi/2, \pi/2]$ to minimize the required order of the Taylor polynomial. The inverse factorial can be calculated in advance and using the Horner scheme for the evaluation of the polynomial, only $K-1$ additions and multiplications are required for each Taylor series. The standard Matlab sin and sinh implementations and the single and double precision Taylor approximations were compared for the calculation of a Hammerich pulse. The respective maximum error as well as the power normalised mean square error are shown in Fig 3. Based on these results it appears that 7 terms are already sufficient to achieve an error below -100 dB and for more than 10 terms the numerical resolution limit of single precision floating point numbers will be reached.

3. SOUND SYNTHESIS EXAMPLES

3.1. Combination of oscillators

Recursive filters are usually used in subtractive synthesis due to computational constraints but will lead to a frequency-dependent phase shift of each harmonic and a predictable combination of several oscillators without unwanted partial cancellations is difficult. The pulse shaping in BLIT corresponds to a linear-phase FIR lowpass filtering, hence all harmonics are still in phase after the filtering and it is straightforward to add or subtract multiple oscillator outputs in a predictable manner to create more complex spectra.

Let us define a signal $x_i(n)$ that has a fundamental frequency $F_{0i} = i \cdot F_0$ which is an integer multiple of another signal $x(n)$ but both share the same spectral envelope. In this case, the difference between $x(n)$ and $x_i(n)$

$$\begin{aligned} x_D(n) &= x(n) - \frac{x_i(n)}{i} \\ &= d(n) * h_H(n) - \frac{d_i(n)}{i} * h_H(n) \\ &= \left[d(n) - \frac{d_i(n)}{i} \right] * h_H(n) \end{aligned} \tag{14}$$

yields a signal where every i-th harmonic is cancelled. For $i = 2$, which is equivalent to using a bipolar impulse train as source signal [1], a signal $x_O(n)$ with only odd harmonics remains (Fig. 4 b).

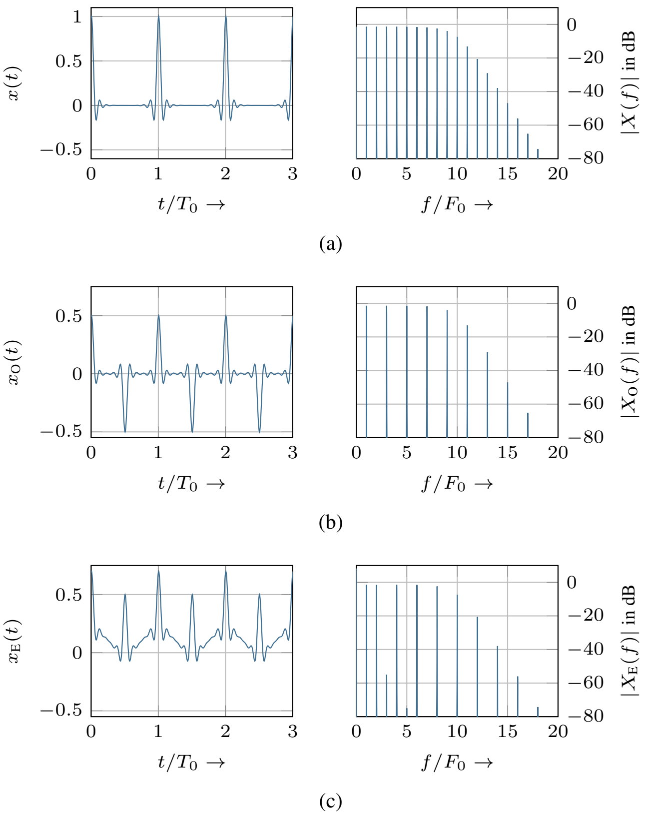


Figure 4: *Unipolar pulse train with full number of harmonics (a), bipolar pulse train with only odd harmonics (b) and signal with even harmonics (c) as a sum of two different unipolar pulse trains.*

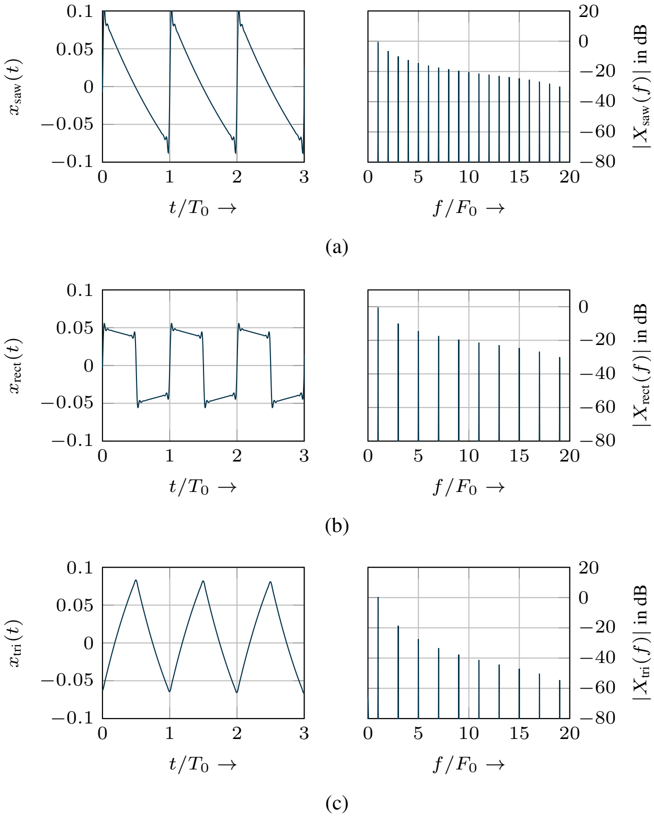


Figure 5: *Sawtooth (a), rectangular (b) and triangular waveforms (c) created by a combination of oscillators and leaky integration.*

Using Hammerich pulses with different parameters for each oscillator offers further possibilities. A signal with even harmonics

$$x_E(n) = x_1(n) + \frac{d_2(n)}{2} * h_H(n), \tag{15}$$

as depicted in Fig. 4 b), can be constructed from the sum of one pulse train $d_2(n)$ with twice the fundamental frequency and arbitrary filter and another single harmonic signal $x_1(n)$ and fundamental frequency F_0.

3.2. Standard waveforms

It was shown in [1] that the standard waveform (rectangular, sawtooth and triangular) can be created with a simple integration of impulse train signals. In our case, a lowpass filtered sawtooth

$$x_{\text{saw}}(n) = x(n) * h_I(n) \tag{16}$$

is obtained by convolving a bandlimited impulse train signal $x(n)$ with an integrator impulse response $h_I(n)$. To avoid accumulation of an error constant in the integration, it is usually recommended to use a leaky integrator. A second order leaky integrator with zero DC gain was proposed by [12]

$$H_I(z) = \pi\frac{\gamma+1}{2}\left(\frac{1-z^{-1}}{1-2\gamma z^{-1}+\gamma^2 z^{-2}}\right) \tag{17}$$

and consists of a cascaded first order leaky integrator and a one-pole highpass. The parameter $\gamma = \exp(2\pi f_{c_I}/f_s)$ defines the cut-off frequency of the highpass (typically $f_{cI} < 20$ Hz) and thereby the crossover point between leaky and non-leaky integration. The rectangular waveform

$$x_{\text{rect}}(n) = x_O(n) * h_I(n) \tag{18}$$

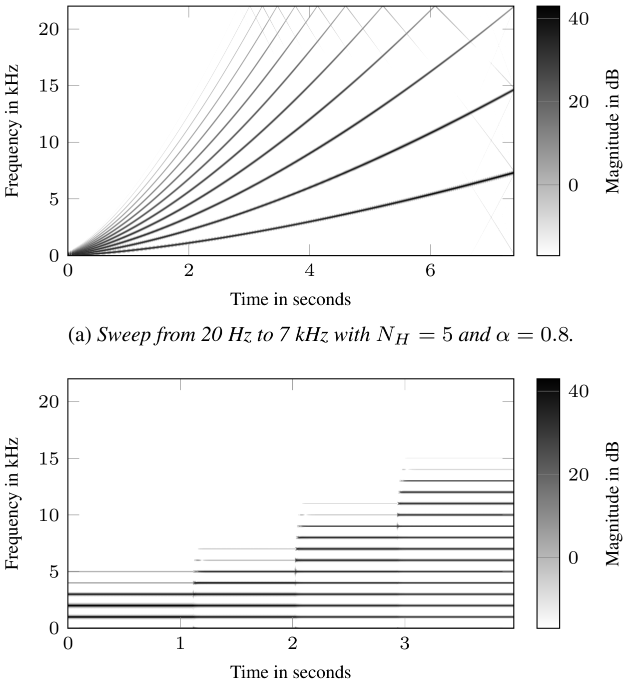
is obtained by leaky integration of a signal with only odd harmonics. Finally, the triangular signal

$$x_{\text{tri}}(n) = x_{\text{rect}}(n) * h_I(n) = x_O(n) * h_I(n) * h_I(n) \tag{19}$$

is an integrated rectangular signal, or two-times integrated signal with odd harmonics. Figure 5 depicts exemplary bandlimited sawtooth, rectangular and triangular signals which were obtained by leaky integration.

3.3. Modulation

All pulse parameters can be directly modulated in a sound synthesis application. Figure 6 a) visualises a fundamental frequency sweep ranging from 20 Hz up to 7 kHz with $N_H = 5$ and $\alpha = 0.8$ at a sample rate of 44.1 kHz. Aliasing is kept at a low level by constantly checking and limiting the parameters N_H and α in dependency of the current fundamental frequency. A step-wise modulation of the number of harmonics is shown in Fig. 6 b).



(a) *Sweep from 20 Hz to 7 kHz with* $N_H = 5$ *and* $\alpha = 0.8$.

(b) *Step-modulated number of harmonics* ($N_H = 3, 5, 9, 13$).

Figure 6: *Example spectrograms showing a modulation of the fundamental frequency and number of harmonics.*

4. CONCLUSION

Usually, in bandlimited impulse train (BLIT) synthesis, sinc-pulses are used to filter a pulse train and to obtain a spectrum with a defined number of harmonics of equal magnitude. In this paper it was proposed to replace the sinc-pulse with a Hammerich pulse as its spectral shape can be directly controlled by two independent parameters for cutoff frequency and filter roll-off. The closed form equation for the Hammerich pulse can be evaluated per sample, does not require the creation of a wavetable and an immediate modulation of the pulse parameters is possible. As all harmonics in a pulse are in phase, differently configured oscillators can be easily combined to create more complex spectral shapes. It was shown how to synthesise spectra with odd or even harmonics and together with a leaky integrator, various standard waveforms (rectangular, triangular, sawtooth) can be created. The Hammerich pulse considerably expands the BLIT principle to become a full-featured synthesis procedure and despite the restriction to lowpass spectra, a wide variety of useful sounds and waveforms can be created without additional filtering.

The possibilities and limitations of this new waveform generation algorithm still have to be explored in practical musical applications. First tests with a real-time modulation of the pulse parameters were quite promising. In particular the simple interface with only a few but very expressive parameters supports an intuitive and creative workflow. For the future it might be in particular interesting to find further pulse shapes which can be calculated and parametrised in a similar fashion as the Hammerich pulse but exhibit a different frequency response, e.g. highpass, bandpass or resonant lowpass.

5. ACKNOWLEDGMENTS

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